Fully Parallel Algorithm for Simulating Nonlinear Schrödinger Equation

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Nonlinear Schrödinger equation in optical fiber communications:

$$iA_z - \frac{1}{2}\beta_2(z)A_{tt} - \frac{i}{6}\beta_3(z)A_{ttt} + \sigma(z)|A|^2 A$$

$$= i(-\gamma + [\exp(z_a\gamma) - 1]\Sigma_{k=1}^{N}\delta(z - z_k))A$$

$$\equiv iG(z)A$$

Amplitude of light: $A\, e^{i(k_0 z - \omega_0 t)}$

β_2 and β_3 - first and second-order group velocity dispersion

$\sigma(z)$ - nonlinear coefficient (Kerr nonlinearity)

$z_k = k z_a \,(k = 1, \ldots, N)$ - amplifiers locations

Change of variables: $u = Ae^{-\int_0^z G(z')dz'}$

$$iu_z - \frac{1}{2}\beta_2(z)u_{tt} - \frac{i}{6}\beta_3(z)u_{ttt} + c(z)|u|^2u = 0$$

$c(z) \equiv \sigma(z)\exp\left(2\int_0^z G(z')dz'\right)$ - periodical function of z

$c(z)$ - depends on fiber nonlinearity, linear losses and amplifiers locations

Generalized nonlinear Schrödinger equation:

$$iu_z \quad + \quad \hat{\mathcal{L}}(z)u \quad + \quad \hat{\mathcal{N}}(|u|^2, z)u = 0$$

Example of generalized nonlinear Schrödinger equation:

$$iu_z \underbrace{- \frac{1}{2}\beta_2(z)u_{tt} - \frac{i}{6}\beta_3(z)u_{ttt}}_{\hat{\mathcal{L}}(z)u} + \underbrace{c(z)|u|^2}_{\hat{\mathcal{N}}(|u|^2, z)u = 0} u = 0$$

Optical fiber communications

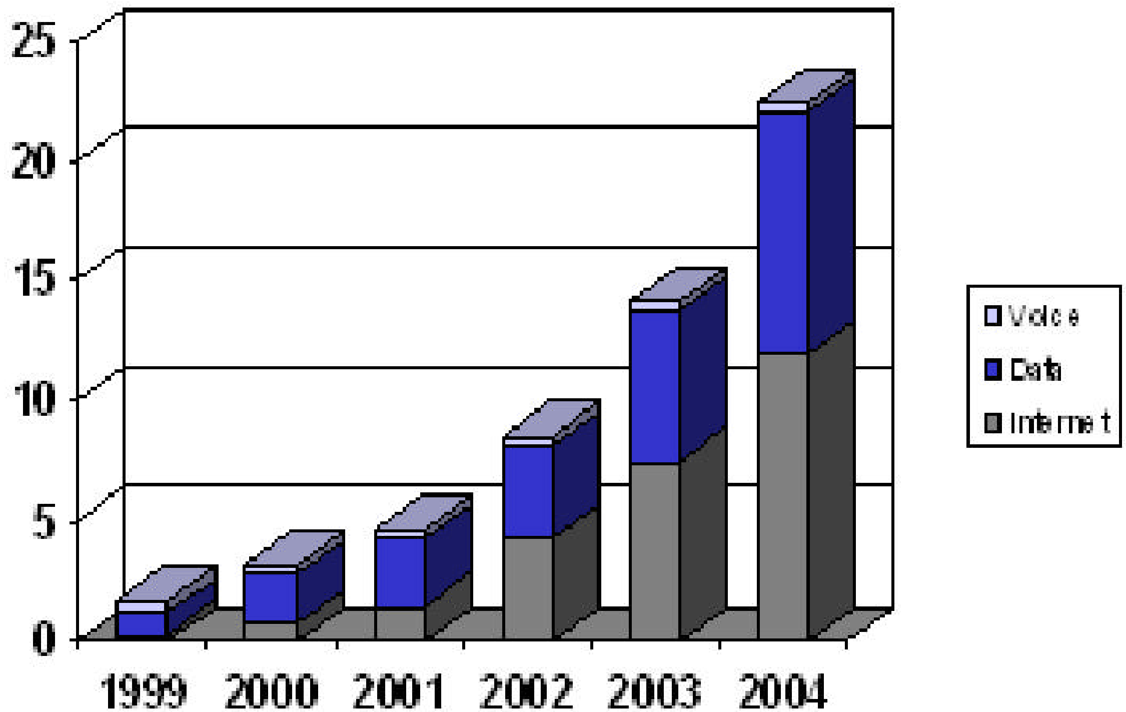
$$iu_z - \frac{1}{2}\beta_2(z)u_{tt} - \frac{i}{6}\beta_3(z)u_{ttt} + c(z)|u|^2 u = 0$$

Amplitude of light: $u\, e^{i(k_0 z - \omega_0 t)}$

β_2 and β_3 - first and second-order group velocity dispersion

$c(z)$ - depends on fiber nonlinearity, linear losses and amplifiers locations

Forecasted Bandwidth Demand – 1999-2004 (Terabits per second)



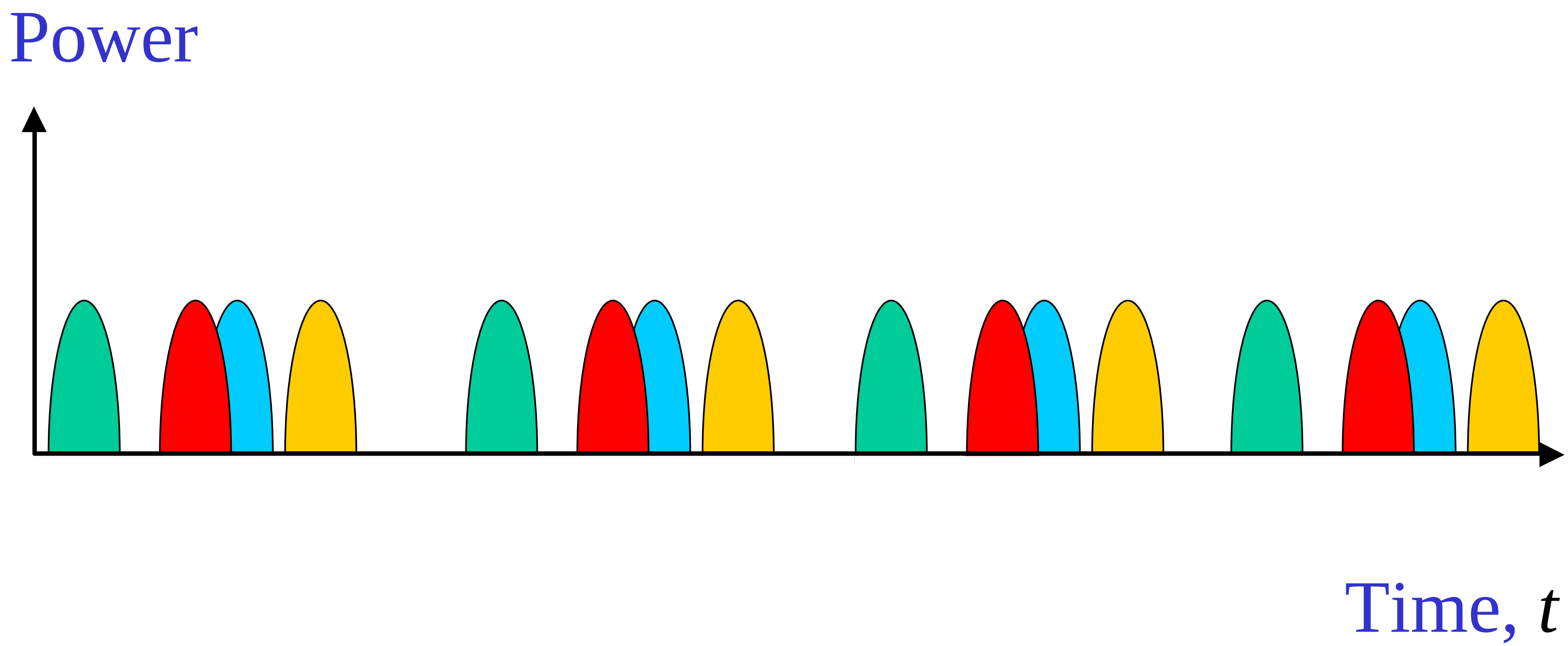
Source: Telegeography, OECD, Renaissance Analysis 1999, RedHerring 2000

Potential solutions to increase bit-rate:

-Increase bit-rate in one frequency channel

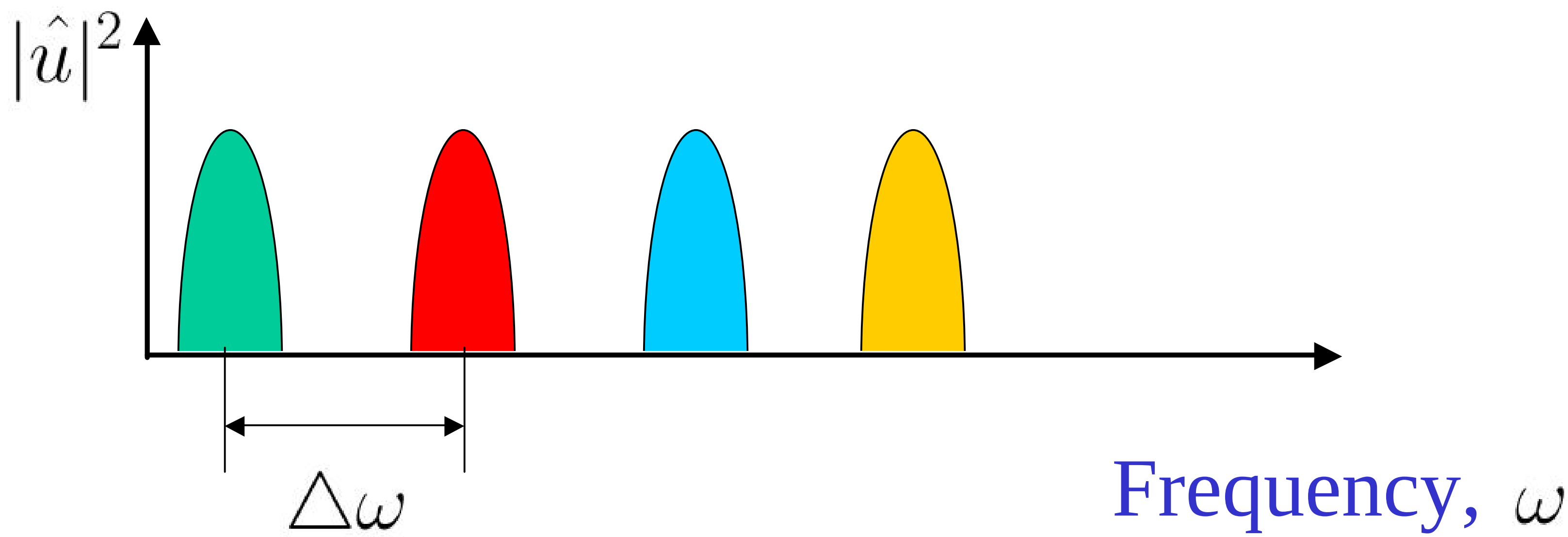
-Increase number of frequency channels

Wavelength-division-multiplexed optical fiber system



Fourier domain

Fourier transform: $\hat{u}(\omega, z) = \int_{-\infty}^{\infty} u(t,z) e^{\imath\omega t} dt.$



$$\Delta\omega/\omega_0 \ll 1$$

Split-step numerical method:

$$\boxed{u_z = \hat{A}u + \hat{B}u}$$

$$u_z = e^{(\hat{A}+\hat{B})z}u(0)$$

$$\simeq e^{\hat{A}z}e^{\hat{B}z}u(0)$$

$\hat{A}$ and $\hat{B}$ do not commute:

$$e^{(\hat{A}+\hat{B})z} = e^{\hat{A}z}e^{\hat{B}z} + O(z^2)$$

$$e^{(\hat{A}+\hat{B})z} = e^{\hat{A}z/2}e^{\hat{B}z}e^{\hat{A}z/2} + O(z^3)$$

Generalized nonlinear Schrödinger equation:

$$\underbrace{iu_z}_{iu_z} \underbrace{- \frac{1}{2}\beta_2(z)u_{tt} - \frac{i}{6}\beta_3(z)u_{ttt}}_{+\;\hat{\mathcal{L}}(z)u} \underbrace{+\; c(z)|u|^2u}_{+\;\hat{\mathcal{N}}(|u|^2, z)u} = 0$$

Split-step:

$$u(z) = e^{i\int_0^z \hat{\mathcal{N}}(|u|^2, z')dz'} e^{i\int_0^z \hat{\mathcal{L}}(z')dz'} u(0) + O(z^2)$$

$\hat{\mathcal{L}}$ is a multiplication operator in Fourier domain:

$$\hat{\mathcal{L}}\hat{u}(\omega, z) = [\beta_2(z)\frac{\omega^2}{2} + \beta_3(z)\frac{\omega^3}{6}]\hat{u}(\omega, z),$$

where $\hat{u}(\omega, z) = \int_{-\infty}^{\infty} u(t, z)e^{\imath\omega t}dt.$

$\hat{\mathcal{N}}$ is a multiplication operator in *t* domain:

$$\hat{\mathcal{N}}u(t, z) = c(z)|u|^2 u(t, z).$$

Estimates of required computation time

Difference in group velocities requires minimal computational window of *1000 bits = 10^5 ps*

How many FFT modes: $\boxed{2\pi\triangle\omega_{max}10^5 ps \sim 2\ 10^5}$

$\triangle\omega_{max} = n\triangle\omega\,,$ n - number of channels

How many steps in *z*: $\triangle z \leq \dfrac{2\pi}{\beta_2\triangle\omega_{max}^2} \sim 3m$

$$\boxed{10^4 km/3m \sim 3\ 10^6}$$

Computation time with split-step: 3 years

Numerical solution of NLS: two steps

$$iu_z - \frac{1}{2}\beta_2(z)u_{tt} - \frac{i}{6}\beta_3(z)u_{ttt} + c(z)|u|^2 u = 0$$

1. Change of variables - weak nonlinearity approximation

2. Effective numerical algorithm for parallel computation

Nonlinear Schrödinger Eq. in Fourier domain:

$$i\hat{u}_z + \hat{\mathcal{L}}\hat{u} + \frac{c(z)}{(2\pi)^2}\int d\omega_1 d\omega_2 d\omega_3 \hat{u}(\omega_1)\hat{u}(\omega_2)\hat{u}^*(\omega_3)$$

$$\times\delta(\omega_1+\omega_2-\omega-\omega_3) = 0\ ,$$

$$\hat{\mathcal{L}}\hat{u}(\omega,z) = [\beta_2(z)\tfrac{\omega^2}{2} + \beta_3(z)\tfrac{\omega^3}{6}]\hat{u}(\omega,z)$$

Change of variables:

$$\hat{u}(\omega,z) \equiv \hat{\psi}(\omega,z)\exp\left(\tfrac{i}{2}\textstyle\int_0^z dz'[\omega^2\beta_2(z') + \tfrac{\omega^3}{3}\beta_3(z')]\right).$$

Integral form of nonlinear Schrödinger Eq.

for new variable $\hat{\psi}$:

$$\hat{\psi}(\omega, z) = \hat{\psi}(\omega, z_0) + iR(\hat{\psi}[\omega, z], \omega, z, z_0),$$

where $$R(\hat{\psi}[\omega, z], \omega, z, z_0) =$$

$$\frac{1}{(2\pi)^2} \int d\omega_1 d\omega_2 d\omega_3 \int_{z_0}^{z} dz' \;\; c(z')\, \hat{u}(\omega_1, z')\hat{u}(\omega_2, z')\hat{u}^*(\omega_3, z')$$

$$\times \exp\Big(-\frac{i}{2} \int_0^{z'} dz''[\omega^2 \beta_2(z'') + \frac{\omega^3}{3}\beta_3(z'')]\Big)\delta(\omega_1 + \omega_2 - \omega - \omega_3),$$

and $$\hat{u}(\omega, z) \equiv \hat{\psi}(\omega, z) \exp\Big(\frac{i}{2} \int_0^{z} dz'[\omega^2 \beta_2(z') + \frac{\omega^3}{3}\beta_3(z')]\Big).$$

Weak nonlinearity approximation

Nonlinear scale: $z_{nl} \gg z_{disp},$

$z_{nl} \equiv 1/|p|^2$ - characteristic nonlinear length,

$z_{disp} \equiv \tau^2/|\beta_2|$ - characteristic dispersion length,

p and τ - typical pulse amplitude and width.

Weak nonlinearity: $\hat{\psi}(\omega, z)$ is a slow function on scale $L \ll z_{nl}$

Fast dependence of $\hat{u}$ is already included in the term $\exp\left(\frac{i}{2}\int_0^z dz'[\omega^2\beta_2(z') + \frac{\omega^3}{3}\beta_3(z')]\right)$ - exact solution of linear part of nonlinear Schrödinger Eq.

First order approximation

$$\hat{\psi}(\omega,(m+1)L) = \hat{\psi}(\omega,mL) +$$

$$iR(\hat{\psi}[\omega,mL],\omega,(m+1)L,mL) + O(\frac{L}{z_{nl}})^2.$$

$\hat{\psi}[\omega,mL]$ replaces $\hat{\psi}[\omega,z]$ in the nonlinear term R in the interval $mL \le z < (m+1)L$.

$O(\frac{L}{z_{nl}})^2$ - accuracy of first order approximation

Second order approximation

$$\hat{\psi}(\omega,(m+1)L) = \hat{\psi}(\omega, mL)$$
$$+iR(\hat{\psi}^{(1)}[\omega, z], \omega, z, mL) + O(\frac{L}{z_{nl}})^3$$

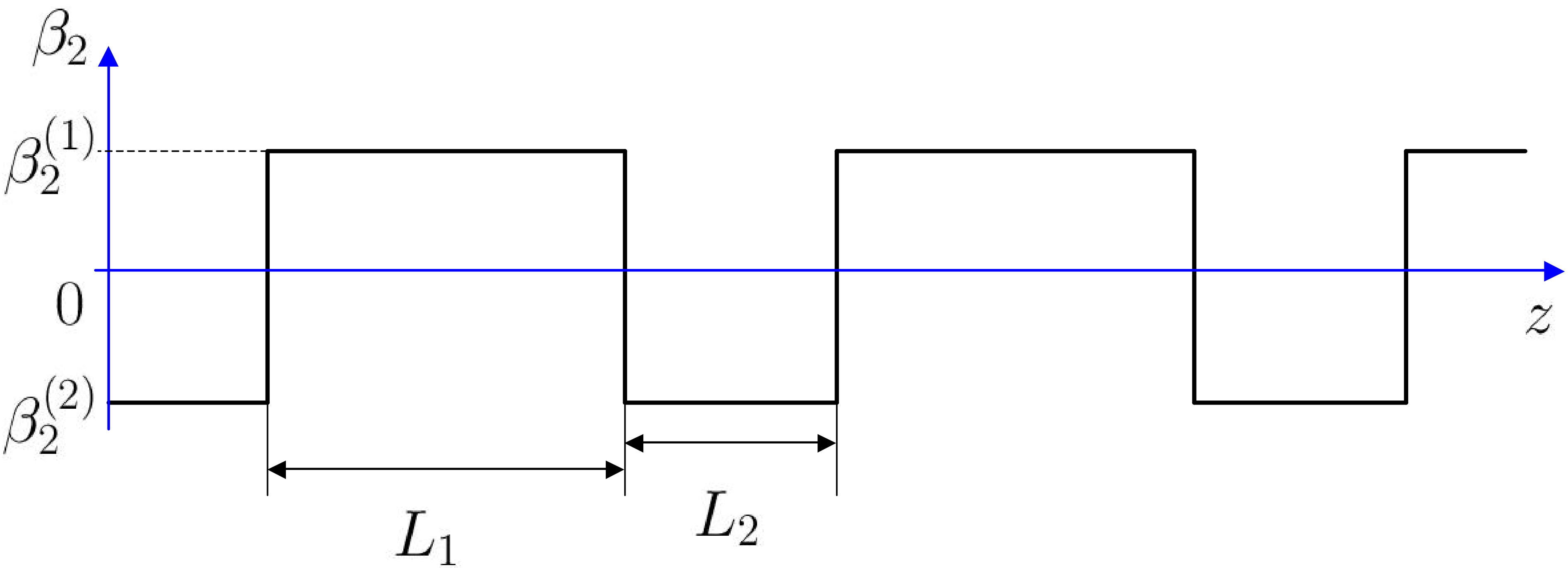
$\hat{\psi}^{(1)}[\omega, z]$ replaces $\hat{\psi}[\omega, z]$ in the nonlinear term R in the interval $mL \leq z < (m+1)L.$

$\hat{\psi}^{(1)}[\omega, z]$ - obtained from first order approximation:

$$\hat{\psi}^{(1)}(\omega, z) \equiv \hat{\psi}(\omega, mL) + iR(\hat{\psi}[\omega, mL], \omega, z, mL).$$

Particular case: path-averaged Gabitov-Turitsyn Eq.

Step-wise dispersion variation:



$L = L_1 + L_2$ - period, $\quad L_1\beta_2^{(1)} + L_2\beta_2^{(2)} = 0$

Nonlinear term:

$$R(\hat{\psi}[\omega, mL], \omega, (m+1)L, mL) = \frac{L}{(2\pi)^2} \int \frac{\sin \frac{s\triangle}{2}}{\frac{s\triangle}{2}} \hat{\psi}(\omega_1, mL)\hat{\psi}(\omega_2, mL)$$

$$\times \hat{\psi}^*(\omega_3, mL)\delta(\omega_1 + \omega_2 - \omega_3 - \omega)d\omega_1 d\omega_2 d\omega_3,$$

$$\triangle \equiv \omega_1^2 + \omega_2^2 - \omega_3^2 - \omega^2$$

$s = L_1\beta_2^{(1)}$ - dispersion map strength

$$\hat{\psi}(\omega, (m+1)L) - \hat{\psi}(\omega, mL) \simeq L\hat{\psi}_z(\omega, mz)|_{z=mL}$$

Gabitov-Turitsyn Eq.:

$$i\hat{\psi}_z(\omega, z) + \frac{1}{(2\pi)^2}\int \frac{\sin\frac{s\Delta}{2}}{\frac{s\Delta}{2}}\hat{\psi}(\omega_1, z)\hat{\psi}(\omega_2, z)\hat{\psi}^*(\omega_3, z)$$
$$\times\delta(\omega_1 + \omega_2 - \omega_3 - \omega)d\omega_1 d\omega_2 d\omega_3 = 0$$

Soliton solution: $\hat{\psi}(\omega, z) = \hat{A}(\omega)e^{i\lambda z}$

λ - soliton propagation constant

Main obstacle of numerical integration of integral equation is a calculation of nonlinear term

$R(\hat{v}[\omega, z], \omega, z, mL)$ - generally requires $M \times N^3$ numerical operations.

N - number of grid points in ω space

M - number of grid points for integration over z

Efficient numerical algorithm for calculation of term $R(\hat{v}[\omega, z], \omega, z, mL)$

Inverse Fourier transform:

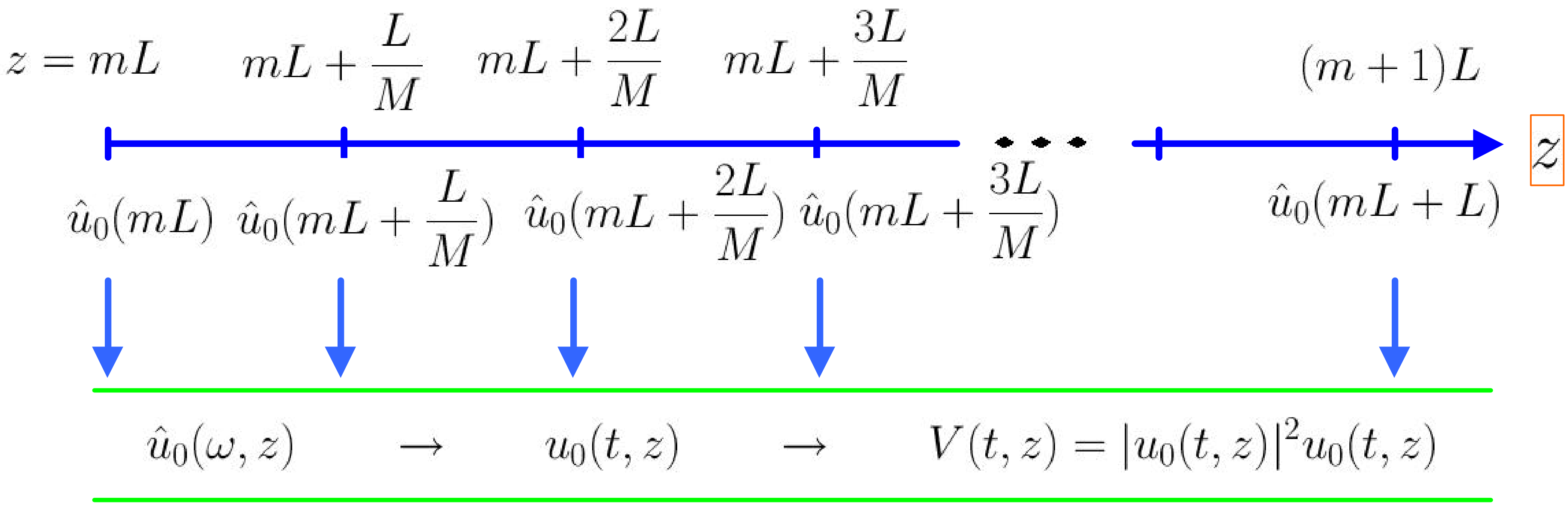
$$\hat{F}^{-1}(R(\hat{v}[\omega], \omega, z, mL)) = \int_{mL}^{z} dz' \, c(z') \mathbf{G}^{(z')}(V^{(z')}(t, z')),$$

$$V^{(z)}(t, z) \equiv |v^{(z)}(t, z)|^2 v^{(z)}(t, z)$$

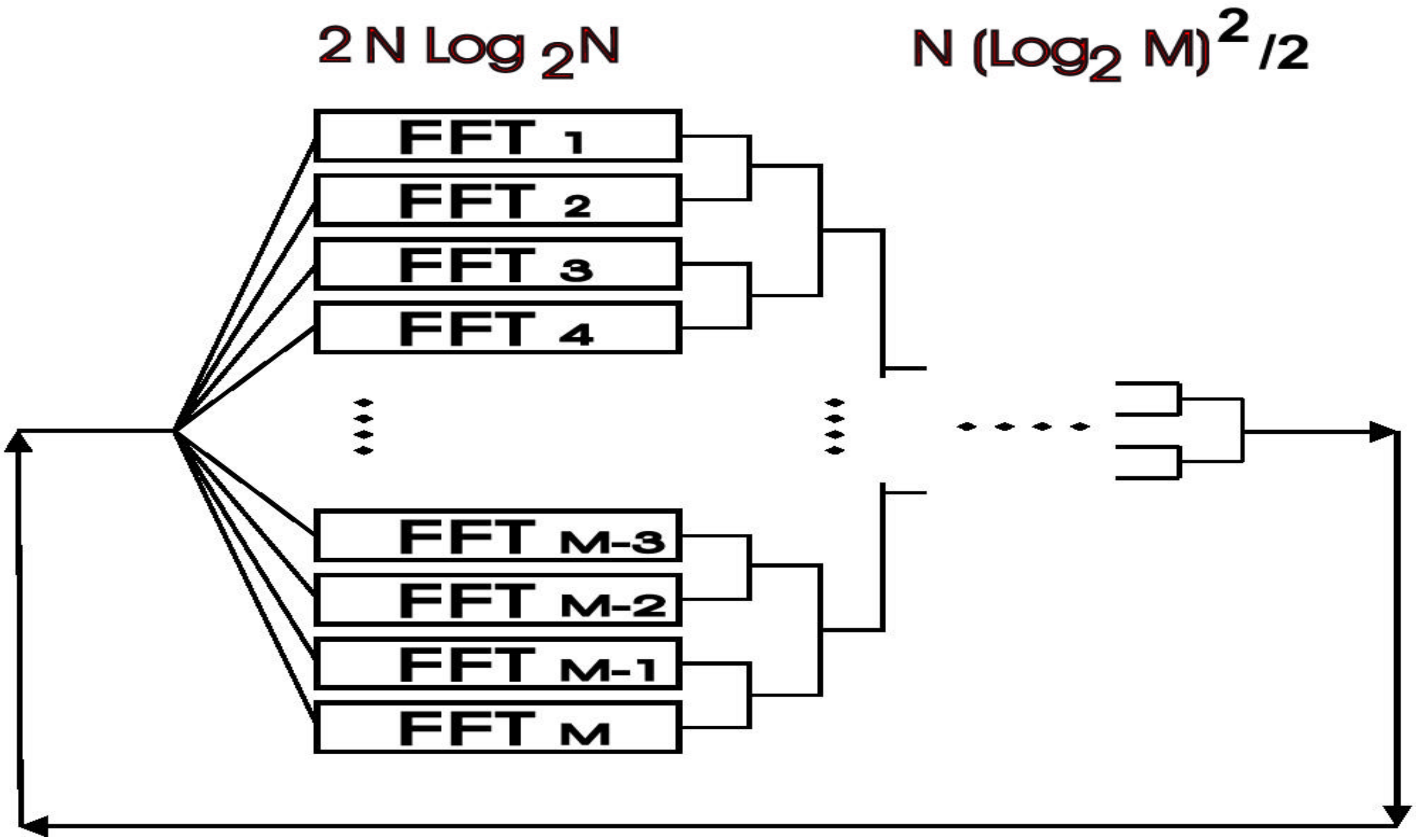
Operator $\mathbf{G}^{(\mathbf{z})}$ is a multiplication operator in ω space: $\hat{\mathbf{G}}^{(z)}(\hat{V}^{(z)}(\omega, z)) \equiv$

$$\exp\left(-\frac{i}{2} \int_0^z dz' [\omega^2 \beta_2(z') + \frac{\omega^3}{3} \beta_3(z')]\right) \hat{V}^{(z)}(\omega, z)$$

$$\boxed{\hat{u}_0(z) = \hat{\psi}(\omega, mL)e^{i\int_{mL}^{z}\hat{\mathcal{L}}(z')dz}}$$



Forward and inverse Fourier transforms, corresponding to M points in z space, can be calculated simultaneously and independently in M CPU's using fast Fourier transform (FFT)



Number of numerical operations:

Split-step: $2MNLog_2(N)$

Parallel algorithm:

$$N[4Log_2(N) + \frac{Log_2(M)^2}{2}]$$

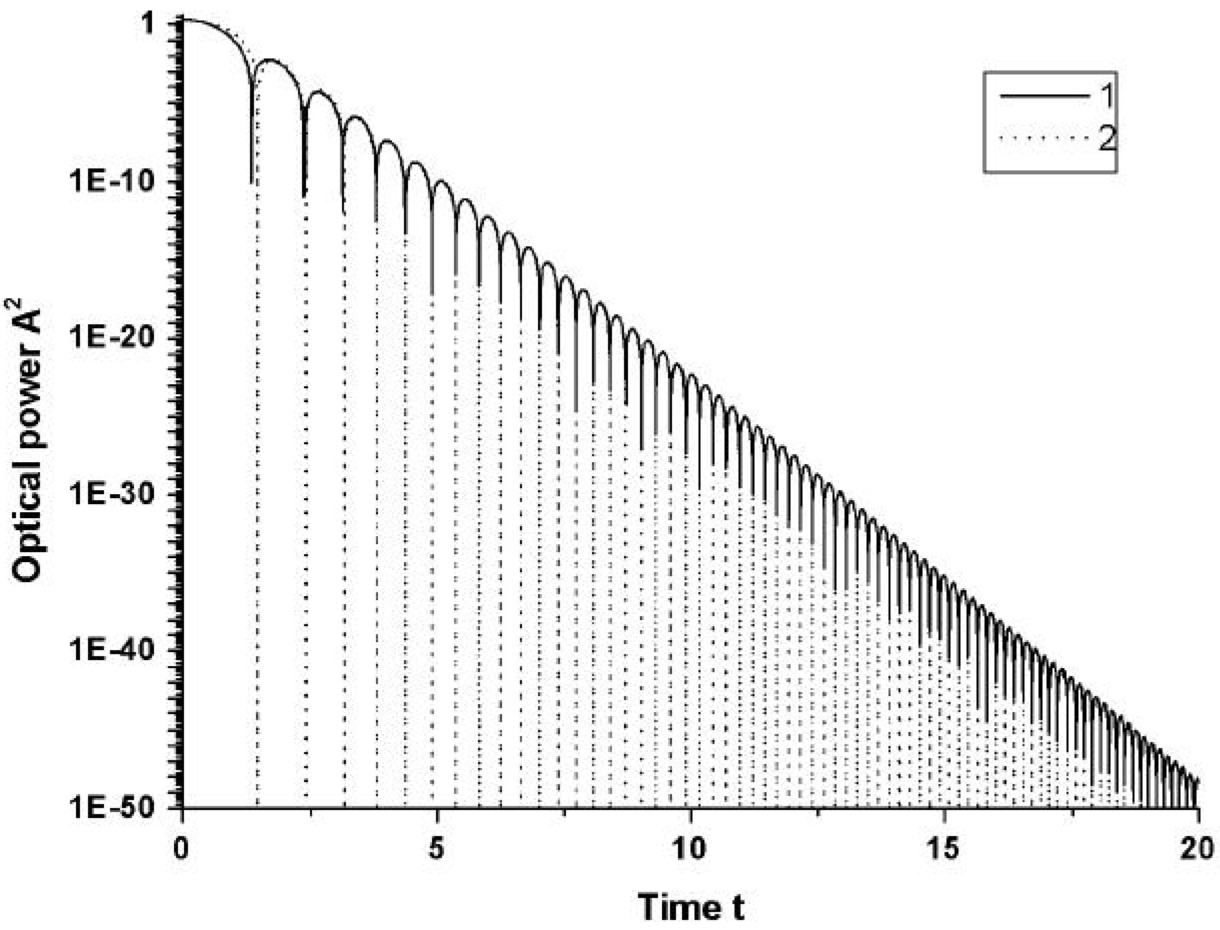
Gabitov-Turitsyn Eq.:

$$i\hat{\psi}_z(\omega, z) + \frac{1}{(2\pi)^2}\int \frac{\sin\frac{s\triangle}{2}}{\frac{s\triangle}{2}}\hat{\psi}(\omega_1, z)\hat{\psi}(\omega_2, z)\hat{\psi}^*(\omega_3, z) \times\delta(\omega_1+\omega_2-\omega_3-\omega)d\omega_1 d\omega_2 d\omega_3 = 0$$

Soliton solution: $\hat{\psi}(\omega, z) = \hat{A}(\omega)e^{i\lambda z}$

λ - soliton propagation constant

Dispersion-managed soliton of Gabitov Turitsyn Eq. versus analytical asymptotic of soliton tails:



Conclusion

The proposed parallel numerical algorithm allows one to implement numerical simulations of NLS about $M/2$ times faster than split-step method.

Future work

Implementation of the proposed massive parallel algorithm on workstation clusters and application for full scale numerical simulations of optical fiber systems.